Final Examination, 9 MAY 1997
SM311O (Spring 1997) – Solutions

The following formulas may be useful to you:

$$a)\ \int\int_S \mathbf{v}\cdot d\mathbf{A} = \int\int\int_D \text{div } \mathbf{v}\, dxdydz, \qquad b)\ \oint_C \mathbf{v}\cdot d\mathbf{r} = \int\int_S \nabla\times\mathbf{v}\cdot d\mathbf{A},$$

$$c)\ \rho(\frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\cdot\mathbf{v}) = -\nabla p + \mu\Delta\mathbf{v} + \rho\mathbf{F}, \qquad \text{div } \mathbf{v} = 0.$$

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Part 1

1. Find the solution to the initial value problem

$$x' = x + y, \quad y' = -x + y, \quad x(0) = 0, y(0) = -2.$$

Solution: In matrix notation, this system is equivalent to

$$\mathbf{x}' = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 0 \\ -2 \end{bmatrix},$$

where the matrix A is

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

We look for solutions of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{e}$. The pair $(\lambda, \mathbf{e})$ forms an eigenvalue-eigenvector pair of matrix A. The eigenvalues of A are $1 \pm i$ with eigenvectors

$$\begin{bmatrix} i \\ 1 \end{bmatrix}, \quad \begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Next, we construct two real linearly independent solutions from one of the eigenvalue-eigenvector pair, say,

$$\mathbf{x}(t) = e^{(1+i)t}\begin{bmatrix} -i \\ 1 \end{bmatrix}.$$

Applying Euler's formula to each term in this solution leads to the general solution

$$\mathbf{x}(t) = c_1 e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + c_2 e^t \begin{bmatrix} \cos t \\ \sin t \end{bmatrix}.$$

Finally, we apply the initial data to determine c_1 and c_2: $c_1 = 0$ and $c_2 = 2$ so

$$\mathbf{x}(t) = 2e^t \begin{bmatrix} \sin t \\ \cos t \end{bmatrix},$$

or $x(t) = 2e^t \sin t$ and $y(t) = 2e^t \cos t$.

2. Find the solution to the initial-boundary value problem

$$u_t = 4u_{xx}, \quad u(0,t) = u(\pi,t) = 0, \quad u(x,0) = 3\sin x.$$

Solution: $u(x,t) = 3e^{4t}\sin x$.

3. (a) Let f be a function of two variables. Describe the geometric relationship between the gradient and the contours of f.

   (b) Let $T(x,y) = x^2 + y^2 - 2x$ be the temperature profile of a two-dimensional body of water, with x and y the coordinates of a typical fluid particle. Draw the graph of the 1-isotherm, i.e., the set of all points that have temperature equal to 1.

   (c) Let $\mathbf{v}$ be a two-dimensional vector field. Define mathematically what it means for $\mathbf{v}$ to have a potential and a stream function. State the necessary conditions (in terms of vector operations) for $\mathbf{v}$ to have a potential and a stream function.

   Solution: 3(c). A vector field $\mathbf{v}$ has a potential ϕ if $\mathbf{v} = \langle \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y} \rangle$. A vector field $\mathbf{v}$ has a stream function ψ if $\mathbf{v} = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle$. $\nabla \times \mathbf{v} = \mathbf{0}$ is a necessary condition for $\mathbf{v}$ to have a potential while div $\mathbf{v} = 0$ is a necessary condition for $\mathbf{v}$ to have a stream function.

   (d) Let $\mathbf{v}$ be a two-dimensional vector field with a potential ϕ and a stream function ψ. Show that the contours of ϕ and ψ must be orthogonal to each other.

   Solution: To show that contours of ϕ and ψ are orthogonal is equivalent to showing that the dot product of $\nabla\phi$ and $\nabla\psi$ is zero (why?). Note, however, that because $\mathbf{v}$ has a potential ϕ, then the following relation holds:

$$\mathbf{v} = \langle \frac{\partial phi}{\partial x}, \frac{\partial phi}{\partial y} \rangle. \tag{1}$$

   On the other hand, $\mathbf{v}$ has a stream function ψ so the following relation holds:

$$\mathbf{v} = \langle \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \rangle. \tag{2}$$

   It then follows from (1) that $\nabla\phi = \mathbf{v}$ and from (2) that $\nabla\psi = \langle -v_2, v_1 \rangle$ so

$$\nabla\phi \cdot \nabla\psi = \mathbf{v} \cdot \langle -v_2, v_1 \rangle = -v_1 v_2 + v_2 v_1 = 0.$$

   This completes the proof.

4. Let $\mathbf{v} = \langle 4xy^3 - y, -x + 6x^2y^2, 6z \rangle$.

   (a) Does $\mathbf{v}$ have a potential ϕ? If no, explain. If yes, find it.

Solution: If a vector field $\mathbf{v}$ has a potential ϕ, then $\nabla \times \mathbf{v}$ must be the zero vector.:

$$\nabla \times \mathbf{v} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy^3 - y & -x + 6x^2y^2 & 6z \end{bmatrix} = \langle 0, 0, 0 \rangle.$$

Now, let ϕ be defined by $\nabla \phi = \mathbf{v}$. Then

$$\frac{\partial \phi}{\partial x} = 4xy^3 - y, \quad \frac{\partial \phi}{\partial y} = -x + 6x^2y^2, \quad \frac{\partial \phi}{\partial z} = 6z. \tag{3}$$

Integrating the first equation in (3) with respect to x yields

$$\phi = 2x^2y^3 - xy + f(y, z). \tag{4}$$

Differentiating this $\phi$with respect to y and comparing the result with the second relation in (3) gives

$$\frac{\partial f}{\partial y} = 0$$

which states that $f(y, z) = g(z)$, some function of z, and, therefore, ϕ in (4) takes the form

$$\phi = 2x^2y^3 - xy + g(z). \tag{5}$$

Differentiating this ϕ with respect to z and comparing the result with the third relation in (3) yields

$$g'(z) = 6z$$

or $g(z) = 3z^2$. Substituting the latter expression in (5) yeilds to the final form of ϕ:

$$\phi = 2x^2y^3 - xy + 3z^2. \tag{6}$$

(b) Compute $\int_C \mathbf{v} \cdot d\mathbf{r}$ where C is the straight line connecting $(0, 0, 0)$ with $(1, -1, 2)$.

Solution: Since $\mathbf{v}$ has a potential ϕ, then $\int_C \mathbf{v} \cdot d\mathbf{r} = \phi|_A^B$, where A and B are the endpoints of C. Hence,

$$\int_C \mathbf{v} \cdot d\mathbf{r} = (2x^2y^3 - xy + 3z^2)|_{(0,0,0)}^{(1,-1,2)} = 11.$$

5. Let $\mathbf{v} = \langle x - 2y, 3x - y \rangle$. Show that this vector field has a stream function and proceed to determine it. Apply this stream function to determine the equation for the path traversed by the particle located at position $(1, -2)$ at time 0.

Please turn over

Part 2

6. (a) Write down a parametrization $\mathbf{r}(u, v)$ of S if
   i. S is the plane that passes through the points $(1, -1, 3)$, $(1, 1, 2)$ and $(0, 0, 0)$.
   ii. S is a sphere of radius 3 centered at $(2, -1, 2)$.

   (b) Find a unit normal vector to the surface $z = 3x^2 + 4y^2$ at $P = (1, 2)$.

7. Let $\mathbf{v}(x, y, z) = \langle 0, 0, 2z - 1 \rangle$ be the velocity field of a fluid flow. Find the flux of this flow through the set of points on the surface $z = 1 - x^2 - y^2$ and located in the upper-half space $z > 0$.

8. Use double or triple integrals to compute the volume of the tetrahedron with vertices, $(1, 0, 0)$, $(0, 1, 0)$, $(1, 1, 0)$ and $(1, 1, 3)$.

9. (a) Let $\psi(x, y) = \cosh \pi x \cos \pi y - 2 \sinh \pi x \sin \pi y$ be the stream function of a fluid flow. Find the velocity at $(x, y) = (1, 1)$.

   (b) Let $\mathbf{v} = \langle \frac{y}{\sqrt{x^2+y^2}}, -\frac{x}{\sqrt{x^2+y^2}} \rangle$. Find the vorticity of $\mathbf{v}$ at $(x, y) = (1, 1)$.

10. Let $\mathbf{v}(x, y, z) = \langle 3x^2, -y^2, 0 \rangle$ be the velocity field of a fluid whose density and viscosity are equal to unity. The position of a fluid particle is denoted by (x, y, z).

    (a) Find the acceleration of the particle that occupies $(1, -1, 1)$.

    (b) Verify whether there is a pressure function p such that the pair $(\mathbf{v}, p)$ satisfies the Navier-Stokes equations with the body force $\mathbf{F} = \mathbf{0}$. If these equations are satisfied, what is the associated pressure p?